

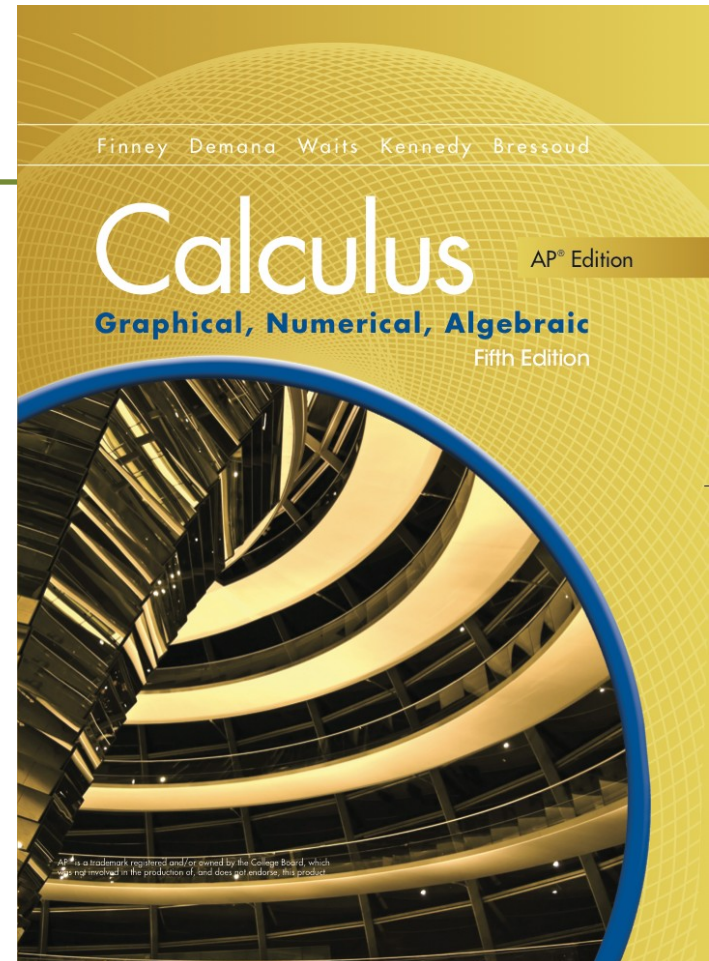
Chapter 9

Sequences, L'Hopital's Rule, and Improper Integrals



Section 9.2

L'Hôpital's Rule



Quick Review

Use graphs or tables to estimate the value of the limit.

1. $\lim_{x \rightarrow 0^-} \left(1 - \frac{1}{x}\right)^x$

2. $\lim_{x \rightarrow -1^+} \left(1 + \frac{1}{x}\right)^x$

3. $\lim_{t \rightarrow 1} \frac{t-1}{\sqrt{t}-1}$

4. $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 - x} + 1}{x+1}$

5. $\lim_{x \rightarrow 0} \frac{\tan x}{2 + \tan x}$

Quick Review

Substitute $x = \frac{1}{h}$ to express y as a function of h

6. $y = x \sin \frac{1}{x}$

7. $y = \left(1 + \frac{1}{x}\right)^x$

Quick Review Solutions

Use graphs or tables to estimate the value of the limit.

$$1. \lim_{x \rightarrow 0} \left(1 - \frac{1}{x}\right)^x \quad 1$$

$$2. \lim_{x \rightarrow -1^-} \left(1 + \frac{1}{x}\right)^x \quad \infty$$

$$3. \lim_{t \rightarrow 1} \frac{t-1}{\sqrt{t}-1} \quad 2$$

$$4. \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 - x} + 1}{x+1} \quad 2$$

$$5. \lim_{x \rightarrow 0} \frac{\tan x}{2 + \tan x} \quad 0$$

Quick Review Solutions

Substitute $x = \frac{1}{h}$ to express y as a function of h

$$6. \ y = x \sin \frac{1}{x} \quad y = \frac{\sinh}{h}$$

$$7. \ y = \left(1 + \frac{1}{x}\right)^x \quad y = (1 + h)^{1/h}$$

What you'll learn about

- L'Hôpital's Rule and its stronger form for repeated iterations
- Indeterminate forms $0/0$ and ∞/∞
- Indeterminate forms $\infty \cdot 0$, and $\infty - \infty$
- Indeterminate forms 1^∞ , 0^0 , ∞^0

...and why

Limits can be used to describe the behavior of functions and l'Hôpital's Rule is an important technique for finding limits.

Indeterminate Form 0/0

If functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ cannot be found by substituting $x = a$. The substitution produces 0/0, a meaningless expression known as an **indeterminate form**.

L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example Indeterminate Form 0/0

Use L'Hôpital's Rule to find the limit.

$$\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x}$$

Let $f(x) = \sqrt{4+x} - 2$ and $g(x) = x$. Since $f(0) = g(0) = 0$ we can apply

L'Hôpital's Rule where $f'(x) = \frac{1}{2}(4+x)^{-\frac{1}{2}}$ and $g'(x) = 1$.

$$\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}(4+x)^{-\frac{1}{2}}}{1} = \frac{1}{4}$$

L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a , and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

Example Using L'Hôpital's Rule with One-Sided Limits

Evaluate using l'Hôpital's Rule: $\lim_{x \rightarrow 0^+} \frac{\sin x}{2x}$

Substituting $x = 0$ leads to the indeterminate form $0/0$. Apply

$$\text{l'Hôpital's Rule: } \lim_{x \rightarrow 0^+} \frac{\sin x}{2x} = \lim_{x \rightarrow 0^+} \frac{\cos x}{2} = \frac{1}{2}$$

Example Working with Indeterminate Form ∞/∞

Identify the indeterminate form and evaluate the limit using

l'Hôpital's Rule. $\lim_{x \rightarrow \pi} \frac{\csc x}{1 + \cot x}$

The numerator and denominator are discontinuous at $x = \pi$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule we can choose I to be any open interval containing $x = \pi$.

$$\lim_{x \rightarrow \pi^-} \frac{\csc x}{1 + \cot x} = \frac{\infty}{\infty}$$

Next differentiate the numerator and denominator.

$$\lim_{x \rightarrow \pi^-} \frac{\csc x}{1 + \cot x} = \lim_{x \rightarrow \pi^-} \frac{-\csc x \cot x}{-\csc^2 x} = \lim_{x \rightarrow \pi^-} \cos x = -1$$

The right hand limit is also equal to -1 . Therefore, the two-sided limit is equal to -1 .

Example Working with Indeterminate Form $\infty \cdot 0$

Find $\lim_{x \rightarrow \infty} \left(2x \sin \frac{1}{x} \right)$.

$\lim_{x \rightarrow \infty} \left(2x \sin \frac{1}{x} \right)$ is of the form $\infty \cdot 0$.

Let $h = \frac{1}{x}$ so $\lim_{x \rightarrow \infty} \left(2x \sin \frac{1}{x} \right) = \lim_{h \rightarrow 0^+} \left(\frac{2}{h} \sin h \right) = 2$

Indeterminate Forms $1^\infty, 0^0, \infty^0$

$$\lim_{x \rightarrow a} \ln f(x) = L \Rightarrow \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} e^{\ln f(x)} = e^L$$

Here a can be finite or infinite.

Example Working with Indeterminate Form 1^∞

Find $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$.

Let $f(x) = \left(1 + \frac{1}{x}\right)^x$. Take the logarithms of both sides:

$$\begin{aligned}\ln f(x) &= \ln \left(1 + \frac{1}{x}\right)^x \\ &= x \ln \left(1 + \frac{1}{x}\right) \\ &= \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}}\end{aligned}$$

This is the indeterminate form $0/0$.

Example Working with Indeterminate Form 1^∞

Apply l'Hôpital's Rule:

$$\lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} = 1$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^1 = e$$

Example Working with Indeterminate Form 0^0

Find $\lim_{x \rightarrow 0^+} x^x$.

The limit is of the indeterminate form 0^0 . Let $f(x) = x^x$ and take the logarithm of both sides. $\ln f(x) = \ln x^x$

$$= x \ln x = \frac{\ln x}{\frac{1}{x}}$$

Apply l'Hôpital's Rule: $\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$

$$= \lim_{x \rightarrow 0^+} \frac{1}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow 0^+} (-x) = 0.$$

Therefore, $\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^0 = 1$

Example Working with Indeterminate Form ∞^0

Find $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

Let $f(x) = x^{\frac{1}{x}}$. Then $\ln f(x) = \frac{\ln x}{x}$.

Apply l'Hôpital's Rule to $\ln f(x)$: $\lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln x}{x}$

$$= \lim_{x \rightarrow \infty} \frac{1}{1} = 0$$

Therefore, $\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$.

Quick Quiz for Sections 9.1 and 9.2

You should solve the following problems without using a graphing calculator.

1. Which of the following gives the value of $\lim_{x \rightarrow 0} \frac{(x+1)^{4/3} - (4/3)x - 1}{x^2}$?

(A) $-1/3$

(B) 0

(C) $2/9$

(D) $4/9$

(E) Does not exist

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(B) 0

(C) $2/9$

(D) $4/9$

(E) Does not exist

Quick Quiz for Sections 9.1 and 9.2

2. Which of the following gives the value of $\lim_{x \rightarrow 0^+} (3x^{2x})$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) Does not exist

Quick Quiz for Sections 9.1 and 9.2

2. Which of the following gives the value of $\lim_{x \rightarrow 0^+} (3x^{2x})$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) Does not exist

Quick Quiz for Sections 9.1 and 9.2

3. Which of the following gives the value of $\lim_{x \rightarrow 2} \frac{\int_2^x \sin t dt}{x^2 - 4}$?

(A) $-\frac{\sin 2}{4}$

(B) $\frac{\sin 2}{4}$

(C) $-\frac{\sin 2}{2}$

(D) $\frac{\sin 2}{2}$

(E) Does not exist

Quick Quiz for Sections 9.1 and 9.2

3. Which of the following gives the value of $\lim_{x \rightarrow 2} \frac{\int_2^x \sin t dt}{x^2 - 4}$?

(A) $-\frac{\sin 2}{4}$

(B) $\frac{\sin 2}{4}$

(C) $-\frac{\sin 2}{2}$

(D) $\frac{\sin 2}{2}$

(E) Does not exist